

3 Complementarity

Name

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Learning objectives

- Define and formulate variational inequality problems
- Define and formulate complementarity problems
- Explain the equivalence between variational inequalities and complementarity problems
- Explain in which cases an optimization problem can be formulated as a complementarity problem and viceversa
- Solve complementarity problems using PATH solver under GAMS

Bibliography

- Basic reading:
 - Facchinei, F. and Jong-Shi P. Finite-dimensional variational inequalities and complementarity problems. Vol. 1. New York: Springer, 2003 (chapter 1).
 - Gabriel, S., Conejo. A., Fuller, D., Hobbs, B., and Ruiz, C. Complementarity modeling in energy markets. Springer, 2012 (chapters 5 and 8).
- Further reading:
 - Cottle, R. W., Jong-Shi Pang, and Richard E. Stone. The linear complementarity problem. Vol. 60. Siam, 2009.

Variational inequalities

Given a set $X \subset \mathbb{R}^n$ and a mapping $F : X \rightarrow \mathbb{R}^n$, a variational inequality problem consists of finding x^* such that

$$(x - x^*)^T F(x^*) \geq 0 \quad \forall x \in X$$

If X is closed and F continuous then the solution to $VI(X, F)$ is a closed set (although it may be empty).

Example 3-1: Relation between Variational inequality and convex optimization problem (I)

Let a convex optimization problem be formulated as

$$\begin{array}{ll} \text{minimize} & f_0(x) \\ \text{subject to} & x \in X \end{array}$$

with f_0 a differentiable function. Write down its first-order optimality condition and compare it with the formulation of a variational inequality.

The first-order optimality condition is

$$(x - x^*)^T \nabla f_0(x^*) \geq 0, \forall x \in X$$

which can be reformulated as a variational inequality problem by defining $F(x) = \nabla f_0(x)$.

Example 3-2: Relation between Variational inequality and convex optimization problem (II)

Let a variational inequality be formulated as

$$(x - x^*)^T F(x^*) \geq 0 \quad \forall x \in X$$

with X a convex set and $F(x) : X \rightarrow \mathbb{R}^n$. Discuss under which conditions this variational inequality can be reformulated as a convex optimization problem.

In order to formulate this variational inequality as an optimization problem we need to find a differentiable function $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that

$$F(x) = \nabla f_0(x)$$

Assuming that you can find such a function $f_0(x)$, this will be convex provided that $\mathbf{J}F(x)$ is a positive semidefinite matrix for all $x \in X$.

✚ Example 3-3: Mapping as gradient of single-valued function (I)

Let a mapping $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be

$$F(x_1, x_2) = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Determine its Jacobian $\mathbf{J}F(x_1, x_2)$ and determine whether there exists a function $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F(x_1, x_2) = \nabla f_0(x_1, x_2)$. *Tip: Try with a quadratic function of the form $f_0(x_1, x_2) = a(x_1)^2 + b(x_2)^2 + cx_1x_2$.*

The Jacobian of F is:

$$\mathbf{J}F(x_1, x_2) = \begin{pmatrix} 4 & 2 \\ 2 & 5 \end{pmatrix}$$

The gradient of the function f_0 is

$$\nabla f_0(x_1, x_2) = \begin{pmatrix} 2ax_1 + cx_2 \\ 2bx_2 + cx_1 \end{pmatrix}$$

Therefore $F(x_1, x_2) = \nabla f_0(x_1, x_2)$ is true for

$$a = 2$$

$$b = 2.5$$

$$c = 2$$

✚ Example 3-4: Mapping as gradient of single-valued function (II)

Let a mapping $F : \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be

$$F(x_1, x_2) = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Determine its Jacobian $\mathbf{J}F(x_1, x_2)$ and determine whether there exists a function $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ such that $F(x_1, x_2) = \nabla f_0(x_1, x_2)$. *Tip: Try with a quadratic function of the form $f_0(x_1, x_2) = a(x_1)^2 + b(x_2)^2 + cx_1x_2$.*

The Jacobian of F is:

$$\mathbf{J}F(x_1, x_2) = \begin{pmatrix} 4 & 3 \\ 2 & 5 \end{pmatrix}$$

The gradient of the function f_0 is

$$\nabla f_0(x_1, x_2) = \begin{pmatrix} 2ax_1 + cx_2 \\ 2bx_2 + cx_1 \end{pmatrix}$$

Therefore $F(x_1, x_2) = \nabla f_0(x_1, x_2)$ is not possible.

Principle of Symmetry

Let $F : X \rightarrow \mathbb{R}^n$ be continuously differentiable on the open convex set $X \subset \mathbb{R}^n$. Then, there exists a real-valued function $f_0 : \mathbb{R}^n \rightarrow \mathbb{R}$ such as $F(x) = \nabla f_0(x)$ if and only if the Jacobian matrix $\mathbf{J}F(x)$ is symmetric for all $x \in X$.

Example 3-5: Are variational inequalities more general than convex optimization problems?

Yes, variational inequalities are more general than convex optimization problems.

Cone

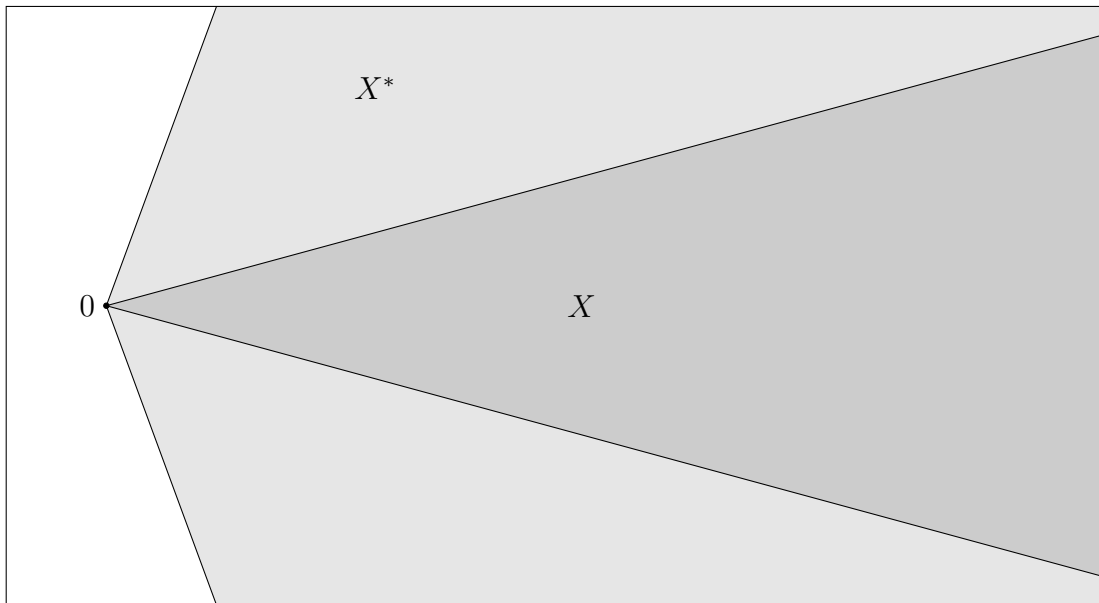
A set X is a cone if for any $x \in X$ and $\theta \geq 0$, then $\theta x \in X$.

Dual cone

Let $X \subset \mathbb{R}^n$ be a cone. Its dual cone is defined as:

$$X^* = \{y \in \mathbb{R}^n : x^T y \geq 0, \forall x \in X\}$$

Example 3-6: Draw a cone and its dual in $\mathbb{R}^2$



Example 3-7: Is the nonnegative orthant $\mathbb{R}_+^n$ a cone? is it convex? What is its dual cone?

The nonnegative orthant $\mathbb{R}_+^n$ is a convex cone and its dual cone is itself.

Complementarity problem

Given a cone $X \subset \mathbb{R}^n$ and a mapping $F : X \rightarrow \mathbb{R}^n$, a complementarity problem consists of finding x^* such that

$$X \ni x^* \perp F(x^*) \in X^*$$

where $x \perp F(x)$ implies that $x^T F(x) = 0$ and X^* is the dual cone of X determined as

$$X^* = \{y \in \mathbb{R}^n : x^T y \geq 0, \forall x \in X\}$$

Relation between complementarity problem and variational inequalities

Let X be a cone and $F : X \rightarrow \mathbb{R}^n$ a mapping. First we assume that x^* solves the $VI(X, F)$, i.e.,

$$(x - x^*)^T F(x^*) \geq 0 \quad \forall x \in X$$

We pick two points $x_1 = 0$ and $x_2 = 2x^*$, which are in X because it is a cone. Therefore we have

$$\left. \begin{aligned} (x_1 - x^*)^T F(x^*) \geq 0 &\implies (0 - x^*)^T F(x^*) \geq 0 \implies -x^{*T} F(x^*) \geq 0 \\ (x_2 - x^*)^T F(x^*) \geq 0 &\implies (2x^* - x^*)^T F(x^*) \geq 0 \implies x^{*T} F(x^*) \geq 0 \end{aligned} \right\} \implies x^{*T} F(x^*) = 0$$

Replacing this in the first equation we obtain

$$(x - x^*)^T F(x^*) \geq 0, \forall x \in X \implies x^T F(x^*) \geq 0, \forall x \in X \implies F(x^*) \in X^*$$

Then proving that x^* also solves $CP(X, F)$. Now we assume that x^* solves $CP(X, F)$. Prove that this implies that x^* also solves $VI(X, F)$.

Since x^* solves $CP(X, F)$ we have that

$$\left. \begin{aligned} x^* &\in X \\ F(x^*) &\in X^* \\ x^{*T} F(x^*) &= 0 \end{aligned} \right\} \implies x^T F(x^*) - x^{*T} F(x^*) = x^T F(x^*) \geq 0, \forall x \in X$$

Non-linear complementarity problem

If X is the nonnegative orthant $\mathbb{R}_+^n$, then we have a non-linear complementarity problem consisting on finding x^* such that

$$0 \leq x^* \perp F(x^*) \geq 0$$

Mixed complementarity problem

If X is the cone $\mathbb{R}^{n_1} \times \mathbb{R}_+^{n_2}$ and $G : \mathbb{R}^{n_1} \times \mathbb{R}_+^{n_2} \rightarrow \mathbb{R}^{n_1}$ and $H : \mathbb{R}^{n_1} \times \mathbb{R}_+^{n_2} \rightarrow \mathbb{R}^{n_2}$ are two mappings, then a mixed complementarity problem consists of finding $(x^*, y^*) \in (\mathbb{R}^{n_1} \times \mathbb{R}_+^{n_2})$ such that

$$\begin{aligned} G(x^*, y^*) &= 0 & x^* \text{ free} \\ 0 &\leq y^* \perp H(x^*, y^*) \geq 0 \end{aligned}$$

Example 3-8: KKT conditions as mixed complementarity problem

Let us assume we have the following convex optimization problem

$$\begin{aligned} &\text{minimize} && f_0(x) \\ &\text{subject to} && f_i(x) \leq 0, \quad i = 1, \dots, m \\ & && h_i(x) = 0, \quad i = 1, \dots, p \end{aligned}$$

Write down its KKT conditions and try to express them as a mixed complementarity problem. The KKT conditions are

$$\begin{aligned} f_i(x^*) &\leq 0, & \forall i = 1, \dots, m \\ h_i(x^*) &= 0, & \forall i = 1, \dots, p \\ \lambda_i^* &\geq 0, & \forall i = 1, \dots, m \\ \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) &= 0 \\ \tilde{\lambda}_i f_i(x^*) &= 0, & \forall i = 1, \dots, m \end{aligned}$$

that can be rewritten as a complementarity problem as follows

$$\begin{aligned} \nabla f_0(x^*) + \sum_{i=1}^m \lambda_i^* \nabla f_i(x^*) + \sum_{i=1}^p \nu_i^* \nabla h_i(x^*) &= 0 & x^* \text{ free} \\ h_i(x^*) &= 0, & \forall i = 1, \dots, p & \nu_i^* \text{ free} \\ 0 &\leq \lambda_i^* \perp -f_i(x^*) \geq 0, & \forall i = 1, \dots, m \end{aligned}$$

📌 Example 3-9: Complementarity formulation for generation scheduling

Formulate the KKT conditions of Example 1-13 as a mixed complementarity problem. The KKT conditions are

$$\begin{aligned}\underline{x} &\leq x \leq \bar{x} \\ \underline{\eta}, \bar{\eta} &\geq 0 \\ 2ax + b - p - \underline{\eta} + \bar{\eta} &= 0 \\ (\underline{x} - x)\underline{\eta} &= 0 \\ (x - \bar{x})\bar{\eta} &= 0\end{aligned}$$

which can be rewritten as a complementarity model as

$$\begin{aligned}2ax + b - p - \underline{\eta} + \bar{\eta} &= 0 & x \text{ free} \\ 0 \leq (x - \underline{x}) \perp \underline{\eta} &\geq 0 \\ 0 \leq (\bar{x} - x) \perp \bar{\eta} &\geq 0\end{aligned}$$

📄 Solving Example 3-9 with GAMS (SchedulingCP.gms)

Solve the complementarity formulation corresponding to Example 1-13 using the same data set and the PATH solver. Compare the results with those previously obtained.

p (€/MWh)	10	20	30	40	50
x (MWh)	0	25	50	75	100

📌 Example 3-10: Complementarity formulation for economic dispatch problem

Formulate the KKT conditions of Example 1-14 as a mixed complementarity problem. The corresponding KKT conditions are

$$\begin{aligned}
 \underline{x}_g &\leq x_g \leq \bar{x}_g, \quad \forall g \\
 \sum_g x_g &= d \\
 \underline{\eta}_g, \bar{\eta}_g &\geq 0, \quad \forall g \\
 2a_g x_g + b_g + \lambda - \underline{\eta}_g + \bar{\eta}_g &= 0, \quad \forall g \\
 (\underline{x}_g - x_g) \underline{\eta}_g &= 0 \\
 (x_g - \bar{x}_g) \bar{\eta}_g &= 0
 \end{aligned}$$

which can be rewritten as a complementarity model as

$$\begin{aligned}
 2a_g x_g + b_g + \lambda - \underline{\eta}_g + \bar{\eta}_g &= 0, \quad x_g \text{ free} \quad \forall g \\
 \sum_g x_g &= d \quad \lambda \text{ free} \\
 0 &\leq (x_g - \underline{x}_g) \perp \underline{\eta}_g \geq 0 \quad \forall g \\
 0 &\leq (\bar{x}_g - x_g) \perp \bar{\eta}_g \geq 0 \quad \forall g
 \end{aligned}$$

📁 Solving Example 3-10 with GAMS (EcoDispatchCP.gms)

Solve the complementarity formulation corresponding to Example 1-14 using the same data set and the PATH solver. Compare the results with those previously obtained.

d (MWh)	100	200	300	400	500
x_{g1} (MWh)	100	200	250	250	250
x_{g2} (MWh)	0	0	50	150	150
x_{g3} (MWh)	0	0	0	0	100